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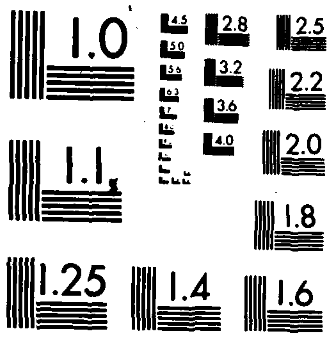

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A CRITICAL EVALUATION OF NEW PLATE THEORIES APPLIED TO LAMINATED COMPOSITES

by

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August 1983

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ABSTRACT

The plate theory recently developed by Levinson is extended to laminates. Closed-form solutions of this theory, as well as those of Reissner-Mindlin plate theory with appropriate shear correction, Seide's discrete-layer plate theory, and Lo, Christensen, and Wu's higher-order theory are all compared with Pagano's elasticity-theory solution for the cases of cylindrical bending of a single orthotropic layer and a symmetric cross-ply (0°/90°/0°) laminate consisting of three equal-thickness layers. Quantities compared are maximum plate deflection, bending stress distribution, and transverse shear stress distribution.

1 INTRODUCTION

It has long been known, through Saint-Venant's flexure theory as well as through experimental observations, that the elementary Bernoulli-Euler beam theory is inaccurate except in the case of pure bending (no transverse shear forces) or very slender geometry (large length/depth). Mindlin and

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Deresiewicz[1] provided an excellent review of early improved theories to take into account transverse shear deformation, including the pioneering work of Bresse[2] in 1859 and analogous work by Timoshenko in 1921-22.[3,4]

A similar situation was evident in the theory of plates, since the classical thin plate theory (CPT) due to Germaine and Lagrange suffers from the same deficiencies of Bernoulli-Euler beam theory, namely

a) Transverse shear strain is neglected

b) In-plane normal strain is distributed <u>linearly</u> through the thickness, rather than nonlinearly

c) Transverse normal strain is neglected

Reissner[5,6] and Mindlin[7] presented generalizations of the Bresse-Timoshenko beam theory to plates and thus made the first attempts to include transverse shear deformation in plate theory. Their theories differed not only in application, References 5,6 to static problems and Reference 7 to dynamic, but also in the definition of the kinematic parameters. These theories not only suffered from deficiencies b and c, listed above, but also the transverse shear strain was distributed linearly through the thickness. This required use of a transverse shear correction factor, either implicitly (Reissner) or explicitly (Mindlin).

Perhaps the first attempts to account for a more realistic distribution of transverse shear strain were due to Ambartsumyan in 1957; see Reference 8, page 40. In his theory, all deficiencies (a,b,c) were removed. Later work conducted in this same spirit was due to Reissner[9] and Levinson[10], and for the geometrically nonlinear case, Schmidt[11]. Numerous so-called high-order theories were presented by Donnell[12], Tiffen[13], Tiffen and Sayer[14], Tiffen and Lowe[15], Lee[16], Berdichevskii[17], Panc[18], Lo et al.[19], Cheng[20], Celep[21],

Krenk[22], Voyiadjis and Baluch[23], and Shirakawa[24]. An interesting tabular comparison of various plate theories was presented by Irretier[25].

The first complete laminated anisotropic plate theory is generally attributed to Reissner and Stavsky[26]; this was a laminated version of CPT. However, it has been known for a long time[27] that transverse shear deformation plays a considerably larger role in structures made of filamentary composite materials than those of isotropic materials. The explanation for this is the very low shear moduli, relative to in-plane elastic moduli, exhibited by such composite materials. The laminated plate versions of the Mindlin and Reissner plate theories are due to Yang et al.[28] and Whitney and Pagano[29], respectively. In Reference 28, a dynamic method of determining the appropriate shear correction factors was introduced and in References 30 and 31, static methods were introduced. Also, the works of Reissner[32,33] Bondar' and Rasskazov[34], and Green and Naghdi[35] should be mentioned.

Higher-order laminated plate theories were reviewed by Lo et al.[36], who applied the laminated version of their own high-order theory[19]. Also, the work of Whitney and Sun[37], Librescu[38], Murthy[39], Soni and Pagano[40], and Rehfield and Vallisetty[41] should be mentioned.

It is noted that in all of the laminated theories discussed above, the shear angle was either not permitted to vary at all from layer to layer (in the theories of References 28 and 29) or required to vary in a smooth, à priori fashion in the higher-order theories. Apparently, the first attempts to consider each layer in a laminate as a separate beam or plate are due to Reference 42 for the static case and Reference 43 for the dynamic case of multicore sandwich beams[+] and Reference 44 for laminated beams loaded statically.

[+]A sandwich beam is generally understood to be one having two or more relatively stiff, thin layers (called facings) and one or more relatively flexible, thick layers (called cores).

This approach was also used by Ambartsumyan[8], page 75, for thick plates, considering each layer as an Ambartsumyan-theory plate. See also the shell work of Hsu and Wang[45] and the plate work of Seide[46], who considered each layer as a Reissner plate.

Here, the terminology "smeared laminate model" (SLM) is used to describe laminate theories of the type (References 28-29, 32-36); "discrete layer model" (DLM) is used to describe laminate theories of the type (References 42-46).

2 METHODOLOGY OF COMPARISON

The objective of an improved theory for laminated plates is to achieve greater accuracy of prediction than is possible with classical thin plate theory (CPT) or even classical (Reissner-Mindlin) shear deformable plate theory (SDT), without requiring the complexity of three-dimensional elasticity theory or even that of the more complicated higher-order plate theories. It is customary to evaluate the accuracy of various improved theories by comparison of the results for a specific situation with those of a three-dimensional elasticity solution for the same situation. In this regard, the closed-form solution due to Pagano[47], for cylindrical bending of a simply supported laminate under a sinusoidally distributed normal pressure loading, has been widely used.

In the present work, four different theories, described in the ensuing, are applied to cylindrical bending in two different cases:

Case 1 Homogeneous orthotropic material

Case 2 Symmetric cross-ply laminate (three layers)

In both cases, the material considered is the same as that of Pagano[47]: $E_L/E_T = 25$, $E_L/G_{LT} = 50$, $G_{LT}/G_{TT} = 2.5$, $\nu_{LT} = \nu_{TT} = 0.25$.

3 THEORIES COMPARED

Classical Shear Deformable Plate Theory (Smeared Laminate Model)

This theory is due to Yang, Norris, and Stavsky[28] and to Whitney and Pagano[29]. It is used here in conjunction with the general shear correction factor derivation given in Reference 31 or more explicitly in Reference 48.

Classical Shear Deformable Plate Theory (Discrete Layer Model)

This theory is due to Seide[46], who applied it and worked it out in detail for a symmetric three-ply laminate.

Laminated Version of Levinson's Theory (Smeared Laminate Model)

This theory is presented in Appendix A. Due to the nonlinearity of the axial-normal-stress distribution through the thickness, use of the equilibrium equation for the xz plane requires a higher degree of nonlinearity in the shear-strain distribution than is assumed *à priori* in this theory. Thus, contrary to the remarks in Reference 19, a shear correction factor may still be needed in this theory. However, due to the complexity of the shear-strain distribution resulting and the dependency of k upon the normal pressure (see Appendix B), it is not practicable to work out this correction factor in general. However, it is shown that for the homogeneous case and $p = 0$ that $k = 1$.

Lo-Christensen-Wu (LCW) Higher-Order Theory (Smeared Laminated Model)

This theory was presented in Reference 19 for the homogeneous case and in Reference 36 for the laminated one.

4 RESULTS AND DISCUSSION

Homogeneous Case

In this case, due to the absence of bending-stretching coupling, $B = C = 0$

and furthermore, E/D = 1/5. Then, eqn (A-15) gives for the Levinson theory

$$w_{max} = (p_o/D\alpha^4) + (6/5)(p_o/Q_{55}h\alpha^2) \quad (1)$$

This result is identical to that of classical Reissner type theory[5,6], i.e., Whitney-Pagano[29] with a shear correction factor k = 5/6.

For comparison with Pagano's numerical results[47], eqn (1) can be rewritten as follows:

$$\bar{w} \equiv 100\,E_T h^3 w_{max}/p_o \ell^4 = (100/\pi^4)[12(1 - \nu_{LT}\nu_{TL}) + (6\pi^2/5)(E_L/G_{LT})(h/\ell)^2](E_T/E_L) \quad (2)$$

Using the previously mentioned material-property ratios in eqn (2), one obtains $\bar{w} = 1.981$ for $h/\ell = 1/4$, considered by the present investigator to be the maximum thickness of a plate rather than a block. This value compares very favorably with a value of approximately 1.95 read from the curve of Reference 47. To the small scale of the plot in Reference 19, this is in agreement with the LCW higher-order theory.

Again for comparison with Reference 47, eqn (A-16) can be used to obtain

$$(\bar{\sigma}_x)_{max} \equiv (\sigma_x)_{max}/p_o = (6/\pi^2)(\ell/h)^2 + (1/10)(E_L/G_{LT})(1 - \nu_{LT}\nu_{TL})^{-1} \quad (3)$$

For $h/\ell = 1/4$, eqn (3) gives $(\bar{\sigma}_x)_{max} = 14.74$, which is in fairly good agreement with Pagano's elasticity-theory value (approximately 14.1). The prediction of classical Reissner SDT is

$$(\bar{\sigma}_x)_{max} = (6/\pi^2)(\ell/h)^2 \quad (4)$$

which yields a value of $(\bar{\sigma}_x)_{max} = 9.727$, which obviously is considerably inaccurate. Incidentally, Fig. 2 of Reference 19 appears to be drawn inaccurately in the vicinity of $h/\ell = 1/4$ and thus cannot be used to compare the LCW value with the above.

Three-Layer Cross-Ply Laminate

Again, the absence of bending-stretching coupling causes coupling stiffnesses B and C to vanish. Also, it can be shown that

$$D = (26\ E_L + E_T)(h^3/324)(1 - \nu_{LT}\nu_{TL})^{-1}$$
$$E = (242\ E_L + E_T)(h^3/14{,}580)(1 - \nu_{LT}\nu_{TL})^{-1}, \quad S_{55} = (28G_{LT} + 26G_{TT})(h/81) \tag{5}$$

Then, application of eqn (A-15) gives, for $h/\ell = 1/4$,

$$\bar{w} \equiv 100\ E_T h^3 w_{max}/p_o \ell^4 = 2.630$$

This deflection value is approximately 12.3% lower than the exact value of approximately 3.0 obtained in Reference 47.

To apply SDT, the following equation for the shear correction factor is used (References 31,48)

$$k = D^2/\{\int G^{(k)} dz \int [b^2/G^{(k)}]\ dz\} \tag{6}$$

or, for the specific laminate (three equal-thickness plies) (Reference 48)

$$k = \frac{5}{6}\ \frac{(26e + 1)^2 g}{(1 + 2g)[102e^2 + g(120e^2 + 20e + 1)]} \tag{7}$$

where $e \equiv E_L/E_T$ and $g \equiv G_{LT}/G_{TT}$. For $e = 25$ and $g = 2.5$, eqn (7) yields $k = 0.5828$. Also, S_{55} and D are given by

$$S_{55} = \int G^{(k)}\ dz = (2\ G_{LT} + G_{TT})(h/3)\ , \quad D = \int z\ Q_{11}^{(k)} dz = \frac{26\ E_L + E_T}{324(1 - \nu_{LT}\nu_{TL})}\ h^3 \tag{8}$$

Then, the maximum deflection is given by

$$w_{max} = (p_o/D\alpha^4) + (p_o/kS_{55}\alpha^2) \tag{9}$$

The result, for $h/\ell = 1/4$, $\bar{w} = 3.226$, which is 7.5% higher than the exact one. A comparison of the present deflection results, for $h/\ell = 1/5$, with those of a number of other investigators is given in Table 1. The bending-stress and shear-stress distributions are shown in Figures 1 and 2.

5 CONCLUDING REMARKS

On the basis of the comparisons made in this study, it is concluded that the Levinson-type theory is more accurate, i.e., closer to the exact elasticity solutions, than classical SDT. The higher-order LCW theory is also more accurate, but requires too much computation to justify the accuracy achieved. Both of these theories predict the nonlinear distribution of bending stress through the thickness, while SDT as well as CPT does not.

However, the Seide theory, which is a discrete layer version of classical SDT, is more accurate in predicting shear-stress distribution than any of the smeared laminate theories mentioned above. Unfortunately, the Seide theory is not quite as accurate in predicting the maximum bending stress, since it predicts a linear distribution of bending stress. This suggests that a new theory, a discrete layer version of the Levinson theory, should be most accurate. It would be expected to have an accurate prediction of shear-stress distribution like the Seide theory and an accurate prediction of the nonlinear bending stress distribution like the Levinson and LCW theories.

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APPENDIX A: LAMINATED VERSION OF LEVINSON PLATE THEORY

Taking Cartesian coordinates x and y in the plane of the plate and z as the thickness normal coordinate, measured positive downward from the midplane of the laminate, one starts with the following displacement field

$$
\begin{aligned}
U(x,y,z) &= u_o(x,y) + z\psi_x(x,y) + z^3\phi_x(x,y) \\
V(x,y,z) &= v_o(x,y) + z\psi_y(x,y) + z^3\phi_y(x,y) \;;\quad W(x,y,z) = w(x,y)
\end{aligned}
\tag{A-1}
$$

It is noted that the midplane displacements, not included in Levinson's original theory[10], are necessary here in order to provide for the bending-stretching coupling exhibited in unsymmetric laminates. Further, as was pointed out by Murthy[32], terms in z^2 are not needed due to the requirement of zero shear stress (and thus zero shear strain) on the upper and lower surfaces of the laminate.

The thickness shear strains are

$$
\gamma_{xz} = U_{,z} + W_{,x} = \psi_x + 3z^2\phi_x + w_{,x} \;;\quad \gamma_{yz} = V_{,z} + W_{,y} = \psi_y + 3z^2\phi_y + w_{,y} \tag{A-2}
$$

where $(\;)_{,x} \equiv \partial(\;)/\partial x$, etc.

Imposing the condition of zero shear strain at the laminate surfaces ($z = \pm h/2$), one can express ϕ_x and ϕ_y in terms of some of the other kinematic variables;

$$
\phi_x = -(4/3h^2)(\psi_x + w_{,x}) \;,\quad \phi_y = -(4/3h^2)(\psi_y + w_{,y}) \tag{A-3}
$$

Thus, the strain field is

$$
\begin{aligned}
\epsilon_x &= U_{,x} = u_{o,x} + z\psi_{x,x} - (4/3h^2)z^3(\psi_{x,x} + w_{,xx}) \\
\epsilon_y &= V_{,y} = v_{o,y} + z\psi_{y,y} - (4/3h^2)z^3(\psi_{y,y} + w_{,yy})
\end{aligned}
$$

$$\gamma_{xy} = U_{,y} + V_{,x} = u_{o,y} + v_{o,y} + z(\psi_{y,x} + \psi_{x,y}) \; ; \; \gamma_{zx} = [1 - (2z/h)^2](\psi_x + w_{,x})$$

$$\gamma_{yz} = [1 - (2z/h^2)](\psi_y + w_{,y}) \quad ; \quad \varepsilon_z = 0 \tag{A-4}$$

Each layer may be monoclinic, i.e., the generalized Hooke's law is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & Q_{16} \\ Q_{12} & Q_{22} & 0 & 0 & Q_{26} \\ 0 & 0 & Q_{44} & Q_{45} & 0 \\ 0 & 0 & Q_{45} & Q_{55} & 0 \\ Q_{16} & Q_{26} & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} \tag{A-5}$$

Here, the Q_{ij} are the plane-stress-reduced stiffness coefficients, σ's are normal stresses, and τ's are the shear stresses.

The plate stress resultants and stress couples are defined as

$$(N_i, M_i) = \int_{-h/2}^{h/2} (1,z)\sigma_i \, dz \qquad (i,j=x,y)$$

$$(N_{xy}, M_{xy}) = \int_{-h/2}^{h/2} (1,z)\tau_{xy} \, dz \tag{A-6}$$

$$(V_x, V_y) = \int_{-h/2}^{h/2} (\tau_{zx}, \tau_{yz}) \, dz$$

Substituting the generalized Hooke's law, eqn (A-5), into eqns (A-6), one obtains

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [A_{ij}] \begin{Bmatrix} u_{o,x} \\ v_{o,y} \\ u_{o,y} + v_{o,x} \end{Bmatrix} + [B'_{ij}] \begin{Bmatrix} \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \end{Bmatrix} + [C_{ij}] \begin{Bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{Bmatrix}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [B_{ij}]\begin{Bmatrix} u_{o,x} \\ v_{o,y} \\ u_{o,y} + v_{o,x} \end{Bmatrix} + [D'_{ij}]\begin{Bmatrix} \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \end{Bmatrix} + [E_{ij}]\begin{Bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{Bmatrix}$$

$$\begin{Bmatrix} V_y \\ V_x \end{Bmatrix} = \begin{bmatrix} S_{44} & S_{45} \\ S_{45} & S_{55} \end{bmatrix}\begin{Bmatrix} \psi_y + w_{,y} \\ \psi_x + w_{,x} \end{Bmatrix} \quad (i,j=1,2,6) \qquad \text{(A-7)}$$

where

$$\left.\begin{aligned} & B'_{ij} = B_{ij} - C_{ij} \quad , \quad D'_{ij} = D_{ij} - E_{ij} \\ & (A_{ij},B_{ij},D_{ij}) = \int_{-h/2}^{h/2} (1,z,z^2)Q_{ij}\, dz \\ & (C_{ij},E_{ij}) = (4/3h^2) \int_{-h/2}^{h/2} (1,z)z^2\, Q_{ij}\, dz \end{aligned}\right\} (i,j=1,2,6) \qquad \text{(A-8)}$$

$$S_{k\ell} = \int_{-h/2}^{h/2} [1 - (2z/h)^2]Q_{k\ell}\, dz \qquad k,\ell=4,5$$

The equilibrium equations of elasticity are

$$\sigma_{ij,j} + F_i = 0 \qquad \text{(A-9)}$$

Integration of eqns (A-9) through the thickness of the laminate and use of eqns (A-6) yield the usual plate equilibrium equations

$$\begin{aligned} & N_{x,x} + N_{xy,y} + p_x = 0 \qquad M_{x,x} + M_{xy,y} - V_x + m_x = 0 \\ & N_{xy,x} + N_{y,y} + p_y = 0 \qquad M_{xy,x} + M_{y,y} - V_y + m_y = 0 \qquad \text{(A-10)} \\ & V_{x,x} + V_{y,y} + p = 0 \end{aligned}$$

where p_i and m_i are body forces and body moments, and p is the normal pressure.

Substitution of the laminate constitutive eqns (A-7) into eqns (A-10) yields finally five plate equilibrium equations in terms of the five generalized displacements u_o, v_o, w_o, ψ_x, ψ_y.

For the case of cylindrical bending, all derivatives with respect to y vanish and the equilibrium equations become

$$A_{11}u_{o,xx} + B'_{11}\psi_{x,xx} - C_{11}w_{,xxx} = 0$$

$$S_{55}(\psi_{x,x} + w_{,xx}) + p = 0 \tag{A-11}$$

$$B_{11}u_{o,xx} + D'_{11}\psi_{x,xx} - E_{11}w_{,xxx} - S(\psi_x + w_{,x}) = 0$$

These can be uncoupled to yield

$$(D_{11} - \frac{B_{11}^2}{A_{11}})\ w_{,xxx} = p - (D_{11} - \frac{B_{11}^2}{A_{11}} - \frac{B_{11}C_{11}}{A_{11}} - E_{11})p_{,xx}/S_{55} \tag{A-12}$$

For a sinusoidally distributed pressure

$$p = p_o \sin \alpha x \quad ; \quad \alpha = \pi/\ell \tag{A-13}$$

the deflection, for freely supported edges, is

$$w = w_{max} \sin \alpha x \tag{A-14}$$

where

$$w_{max} = \frac{p_o}{[D - (B^2/A)]\alpha^4} + \frac{p_o}{S_{55}\alpha^2}(1 + \frac{BC - AE}{AD - B^2}) \tag{A-15}$$

and the subscripts 11 have been omitted from A, B, C, D, and E for brevity.

In eqn (A-15), the first term is identical to that of CPT and the quantity

$$1 + (BC - AE)(AD - B^2)$$

is the multiplier of classical shear deformable plate theory.

It is interesting to note that the normal strain is not distributed linearly through the thickness but has a term in the cube of z:

$$\varepsilon_x = [(B/A) - z]w_{,xx} - (\frac{C - B}{A} + z - \frac{4}{3}\frac{z^3}{h^2})(p/S_{55}) \tag{A-16}$$

APPENDIX B: CORRECTION COEFFICIENT FOR A LAMINATE IN LEVINSON THEORY

There are numerous methods of determining the shear correction coefficient, some dynamic and some static. The two most popular static methods are:

1. Through use of the axial-force-equilibrium equation of elasticity. This approach is identical to that of Jourawsky's theory of transverse shear in beams (and Reissner's for plates) and yields a value of 5/6 for a homogeneous rectangular section.

2. Through use of equivalency with the results of Saint-Venant theory of flexure. This method, originated by Cowper[4], yields a value dependent upon Poisson's ratio for a homogeneous rectangular section of isotropic material.

Here, the former approach is used. The plane-strain equilibrium of axial forces in the xz plane is expressed by

$$\sigma_{x,x} + \tau_{xz,z} = 0 \qquad \text{(B-1)}$$

Thus,

$$\tau_{xz} = - \int_{-h/2}^{z} \sigma_{x,x} \, dz \qquad \text{(B-2)}$$

However,

$$\gamma_{xz} = \tau_{xz}/Q_{55}^{(k)} \qquad \text{(B-3)}$$

where the superscript k denotes the k^{th} layer.

Thus,

$$\gamma_{xz} = -[1/Q_{55}^{(k)}] \int_{-h/2}^{z} Q_{11}^{(k)} \, \varepsilon_{x,x} \, dz \qquad \text{(B-4)}$$

or

$$\gamma_{xz} = -[1/Q_{55}^{(k)}] \int_{-h/2}^{z} Q_{11}^{(k)}[u_{o,xx} + z\psi_{x,xx} - (4/3h^2)z^3(\psi_{x,xx} + w_{,xxx})]dz \qquad \text{(B-5)}$$

Now, it is necessary to express the generalized displacement in terms of

the generalized forces by inversion of the laminate constitutive relations, eqns (A-7). The results are

$$u_{o,x} = \frac{DN_x - BM_x}{AD - B^2} + \frac{DC - BE}{AD - B^2}(V_{x,x}/S_{55})$$

$$\psi_{x,x} = \frac{AM_x - BN_x}{AD - B^2} + \frac{AE - BC}{AD - B^2}(V_{x,x}/S_{55}) \tag{B-6}$$

$$\psi_{x,x} + w_{,xx} = V_x/S_{55}$$

where subscripts 11 have been omitted from A, B, C, and D for brevity.

Substituting relations (B-6) into eqn (B-5), one obtains

$$\gamma_{xz} = -[1/Q_{55}^{(k)}]\int_{-h/2}^{z} Q_{11}^{(k)}\{\frac{DN_{x,x} - BM_{x,x} + (CD - BE)(V_{x,xx}/S_{55})}{AD - B^2}$$

$$+ \frac{AM_{x,x} - BN_{x,x} + (AE - BC)(V_{x,xx}/S_{55})}{AD - B^2} z - (4/3h^2)(V_{x,x}/S_{55})z^3\}dz \tag{B-7}$$

Now, beam-type equilibrium requires that

$$N_{x,x} = 0 \quad , \quad M_{x,x} = V_x \quad , \quad V_{x,x} = -p \tag{B-8}$$

Thus, eqn (B-7) reduces to

$$\gamma_{xz} = -[1/Q_{55}^{(k)}]\{\frac{Ab - Ba}{AD - B^2} V_x + (cp/S_{55}) + [\frac{(CD - BE)a + (AE - BC)b}{AD - B^2}](-p_{,x}/S_{55})\} \tag{B-9}$$

where the partial stiffnesses are given by

$$(a,b) \equiv \int_{-h/2}^{z} (1,z)Q_{11}^{(k)}\,dz$$

$$c \equiv (4/3h^2)\int_{-h/2}^{z} z^3 Q_{11}^{(k)}\,dz \tag{B-10}$$

For a symmetric laminate, B = C = 0 and eqn (B-10) reduces to

$$\gamma_{xz} = -\,1/Q_{55}^{(k)}[(b/D)V_x + (c/S_{55})p - (Eb/DS_{55})p_{,x}] \tag{B-11}$$

To derive the shear correction factor (k), one uses

$$k\,U_S{}^C = U_S{}^E \tag{B-12}$$

where $U_S{}^C$ and $U_S{}^E$ are the shear strain energies (per unit length) calculated on the basis of the constitutive relation and the equilibrium equation of elasticity, respectively. Thus,

$$U_S{}^C = \frac{1}{2}\int_{-h/2}^{h/2} G(\psi_x + w_{,x} + 3z^2\phi_x)^2 dz = \frac{\gamma_o{}^2}{2}\int_{-h/2}^{h/2} (1 - 4\,\frac{z^2}{h^2})^2 \;G\;dz \tag{B-13}$$

where

$$\psi_x + w_{,x} = \gamma_o \quad , \quad \phi_x = -(4/3h^2)\gamma_o \tag{B-14}$$

Also,

$$U_S{}^E = \frac{1}{2}\int_{-h/2}^{h/2} \frac{1}{G}\,[(b/D)V_x + (c/S_{55})p - (Eb/DS_{55})p_{,x}]^2 dz \tag{B-15}$$

But

$$V_x = S_{55}\gamma_o \tag{B-16}$$

Thus, eqn (B-15) can be rewritten as

$$U_S{}^E = \frac{1}{2}\int_{-h/2}^{h/2} \frac{1}{G}[(bS_{55}/D)\gamma_o + (c/S_{55})p - (Eb/DS_{55})p_{,x}]^2 dz \tag{B-17}$$

Finally, in view of eqn (B-12)

$$k^{-1} = \frac{\gamma_o^2 \int_{-h/2}^{h/2} \left(1 - 4\frac{z^2}{h^2}\right)^2 G\, dz}{\int_{-h/2}^{h/2} \frac{1}{G}\left[(bS_{55}/D)\gamma_o + (c/S_{55})p - (Eb/DS_{55})p_{,x}\right]^2 dz} \tag{B-18}$$

It is noted that, in contrast to the shear correction factor in Bresse-Timoshenko-Reissner theory which is independent of p, the present expression for k depends upon p and $p_{,x}$. Since for the sinusoidal distribution of p, the quantities p and $p_{,x}$ vary with x, eqn (B-18) implies that k must vary with x. However, it is not practicable to consider this variation here. Thus, for $p = p_{,x} = 0$, eqn (B-18) reduces to

$$k^{-1} = \frac{\int_{-h/2}^{h/2} \left(1 - 4\frac{z^2}{h^2}\right)^2 G\, dz}{(S_{55}/D)^2 \int_{-h/2}^{h/2} (b^2/G)\, dz} \tag{B-19}$$

For the homogeneous (single-layer) case, Q_{11} and G are independent of z and

$$b = (1/2)[z^2 - (h^2/4)]\, Q_{11}$$

Then, eqn (B-19) gives k = 1 precisely.

TABLE 1

Dimensionless Deflection of a 0°/90°/0° Laminate by

Various Theories, for $h/\ell = 1/5$

Theory	Reference	$\bar{w} \equiv 100E_T h^3 w_{max}/p_o \ell^4$
Exact	47	$\sim$ 2.0*
Seide	46	$\sim$ 2.0*
Murthy	39	$\sim$ 1.87*
Present (SDT)	-	2.25
Present (Levinson)	-	1.87

*The values marked with the approximation sign ($\sim$) are only approximate, since they were obtained from reading small-size curves.

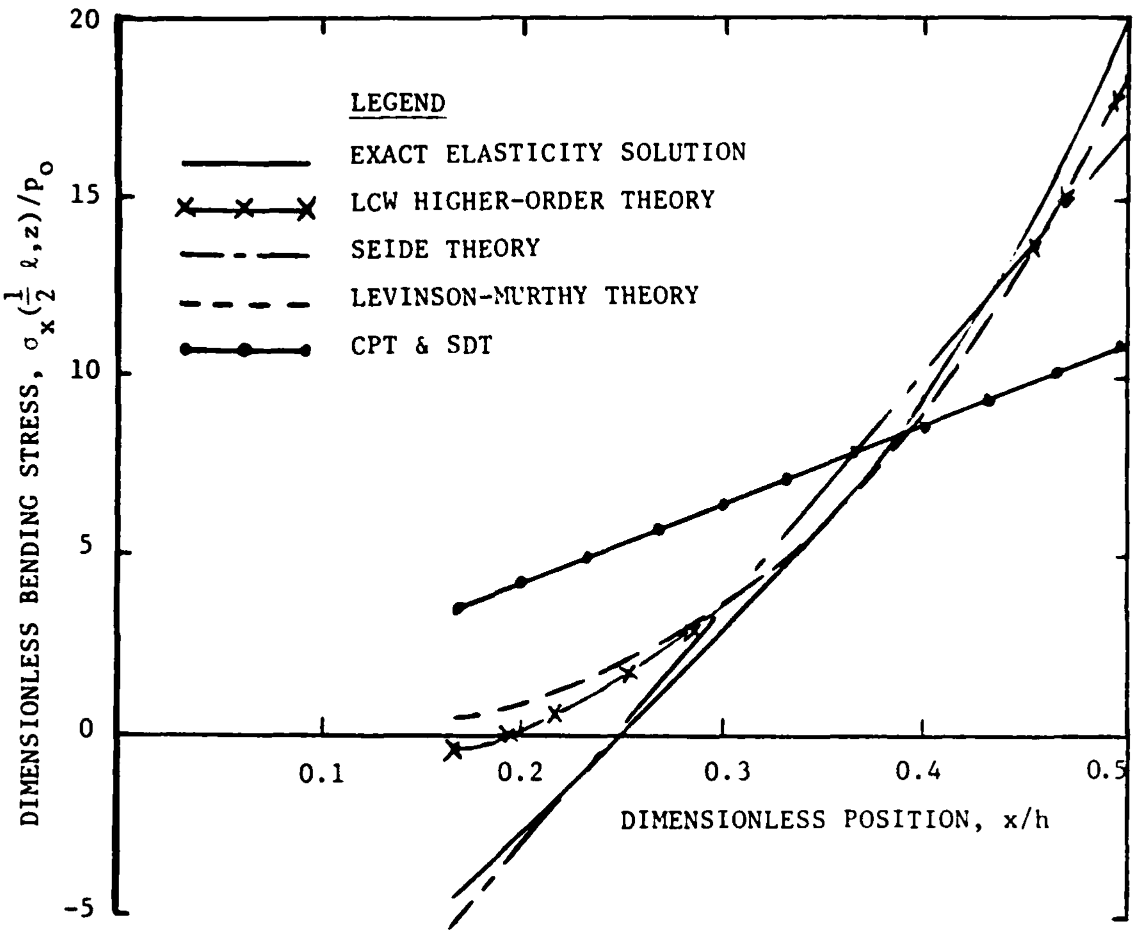

Fig. 1 Bending stress distribution through half thickness of three-ply laminate; $h/\ell = 1/4$. Stress in middle ply too small to show at this scale.

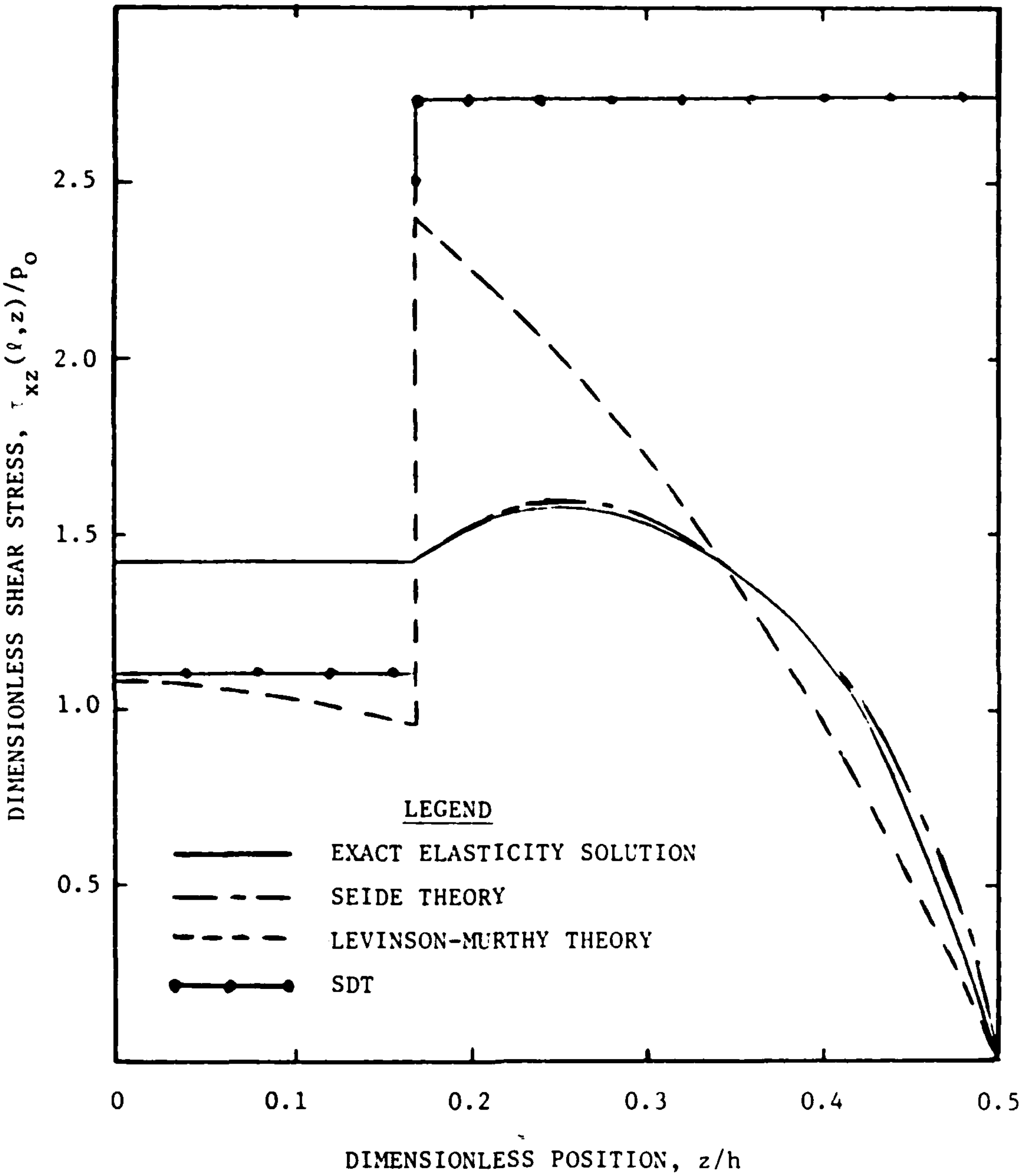

Fig. 2 Shear stress distribution through half thickness of three-ply laminate; $h/\ell = 1/4$.

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30	OU 82-5	Analyses of Beams Constructed of Nonlinear Materials Having Different Behavior in Tension and Compression	C.W. Bert & F. Gordaninejad
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18. SUPPLEMENTARY NOTES

An abbreviated and preliminary version of this paper is to be presented at the Symposium on Mechanics of Composite Materials, sponsored by the Applied Mechanics Division, ASME Winter Annual Meeting, Boston, MA, Nov. 13-18, 1983.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Classical solutions, closed-form solutions, composite materials, cross-ply laminates, cylindrical bending, elasticity theory, higher-order plate theories, laminate theory, laminated plates, moderately thick plates, shear correction factor, shear deformable plate theory, static plate theory, transverse shear deformation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The plate theory recently developed by Levinson is extended to laminates. Closed-form solutions of this theory, as well as those of Reissner-Mindlin plate theory with appropriate shear correction, Seide's discrete-layer plate theory, and Lo, Christensen, and Wu's higher-order theory are all compared with Pagano's elasticity-theory solution for the cases of cylindrical bending of a single orthotropic layer and a symmetric cross-ply (0°/90°/0°) laminate consisting of three equal-thickness layers. Quantities compared are maximum plate deflection, bending stress distribution, and transverse (over)

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20. Abstract - Cont'd

shear stress distribution.

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